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HEAT CONDUCTION AND HEAT TRANSFER IN TECHNOLOGICAL PROCESSES

ANALYTICAL SOLUTIONS OF THE PROBLEMS OF HEAT TRANSFER DURING LIQUID FLOW IN PLANE-PARALLEL CHANNELS BY DETERMINING THE TEMPERATURE PERTURBATION FRONT

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Based on the integral method of heat balance, an analytical solution of the problem of heat transfer in stabilized liquid flow in a plane tube is obtained. To increase the accuracy of solution, the approximation of the temperature function is made by polynomials of higher degrees. To determine their coefficients, supplementary boundary conditions are introduced that are found from the basic differential equation and given boundary conditions including conditions at the temperature perturbation front. In the second approximation the obtained analytical solution in the range of the longitudinal coordinate $10^{-6} \le x \le \infty$ *already virtually coincides with the exact one.*

Introduction. Among approximate analytical methods a special place is occupied by those which employ the notion of the temperature perturbation front (TPF). Among them there are integral methods of heat balance. The process of heating (cooling) of bodies is formally divided here into two stages: 1) a gradual advance of the TPF from the surface to the center, and 2) a change in the temperature over the entire volume of a body up to the onset of a steady-state mode of flow [1–4]. The finite velocity of the advance of TPF is accounted for by introducing a new function $q_1(x)$ called the penetration depth (the depth of a thermal layer). Such a model of the process of heat conduction is used in a number of methods [1–4].

However, the drawback of these methods is the necessity of an a priori choice of the coordinate dependence of the temperature function sought. This ambiguity of the solution raises the problem of accuracy, since, adopting beforehand one or another profile, different results are obtained each time. Moreover, there is a whole class of problems for which this method is of low efficiency: problems with a pulse change in input functions, with a variable initial condition, with local heat sources, etc.

An obvious way of increasing the accuracy of integral methods is approximation of the temperature function by polynomials of higher degrees. However, to determine the unknown coefficients of these polynomials it appears that the initial boundary conditions are insufficient, and supplementary boundary conditions are needed. In the present work such conditions are found from the initial differential equation using basic boundary conditions, including the conditions at the TPF. The use of the initial differential equation in deriving supplementary boundary conditions allows its closing at a minimum number of approximations [5–7].

Statement of the Problem. Let us find an approximate analytical solution of the problem of heat transfer for a liquid moving in a plane tube. We make the following assumptions [8]: the flow of the liquid is stabilized, the process of heat transfer is steady, the liquid is incompressible, its physical properties are constant, and the thermal conductivity of the medium in the direction of its flow is neglected — this assumption is fulfilled at Pe $\ge$ 100, which is valid for gases and nonmetallic liquids [9].

With account for the assumptions made, the mathematical statement of the problem has the form [8]

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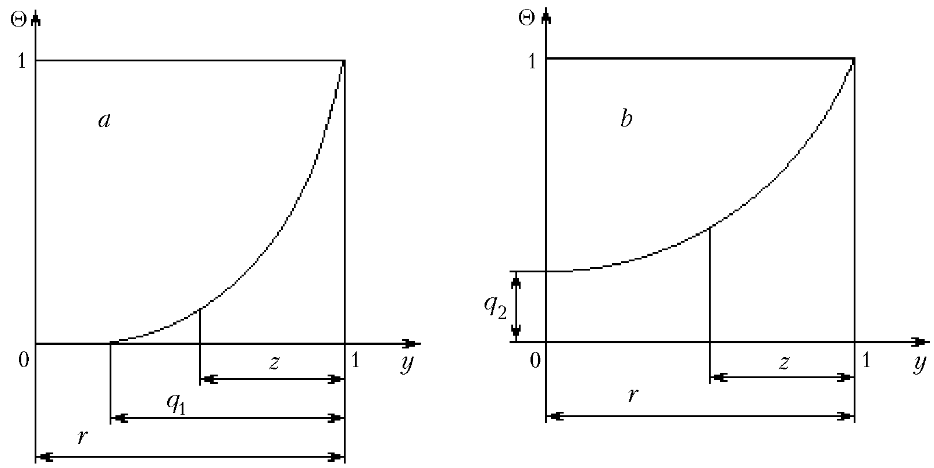

Fig. 1. The scheme of the computation of heat transfer.

$$\omega \frac{\partial t(\xi, \eta)}{\partial \eta} = a \frac{\partial^2 t(\xi, \eta)}{\partial \xi^2}, \quad \eta > 0, \quad 0 \le \xi < r ; \tag{1}$$

$$t(\xi, 0) = t_0 ; \tag{2}$$

$$\frac{\partial t(0, \eta)}{\partial \xi} = 0 ; \tag{3}$$

$$t(r, \eta) = t_w . \tag{4}$$

We will introduce the following nondimensional variables and parameters:

$$\Theta = \frac{t - t_0}{t_w - t_0}, \quad y = \frac{\xi}{r}, \quad x = \frac{8}{3} \frac{1}{\mathrm{Pe}} \frac{\eta}{h}, \quad \mathrm{Pe} = \frac{\omega_{av} r}{a} .$$

With allowance for the notation adopted, the problem, Eqs. (1)–(4), will be represented as follows:

$$(1 - y^2) \frac{\partial \Theta(y, x)}{\partial x} = \frac{\partial^2 \Theta(y, x)}{\partial y^2}, \quad x > 0, \quad 0 \le y < 1 ; \tag{5}$$

$$\Theta(y, 0) = 0 ; \tag{6}$$

$$\frac{\partial \Theta(0, x)}{\partial y} = 0 ; \tag{7}$$

$$\Theta(1, x) = 1 . \tag{8}$$

The process of heating of the medium will be divided into two stages over the coordinate x, that is, $0 \le x \le x_1$ and $x_1 \le x < \infty$, and for this purpose we introduce a boundary that moves along the coordinate y and that divides the initial region into two subregions: heated $0 \le y \le q_1(x)$ and unheated $q_1(x) \le y \le 1$. Here $q_1(x)$ is a function that determines the advance of the interface along the coordinate y depending on the longitudinal coordinate x (Fig. 1a). Moreover, in the region not affected by heating, the temperature t_0 given at the channel inlet is preserved. The first stage of the process ends when the moving interface reaches the middle of the channel. In the second stage of heat transfer the temperature changes over the entire volume of the medium $0 \le y < 1$ (Fig. 1b).

If we pass from the coordinate y, reckoned from the center of the channel, to a new variable $z = 1 - y$ reckoned from the surface, the problem of the heating of the liquid for the first stage of the process can be formulated in the form

$$(2z - z^2) \frac{\partial \Theta (z, x)}{\partial x} = \frac{\partial^2 \Theta (z, x)}{\partial z^2} , \quad 0 \le x \le x_1 , \quad 0 \le z \le q_1 (x) ; \tag{9}$$

$$\Theta (z, 0) = 0 ; \tag{10}$$

$$\Theta (0, x) = 1 . \tag{11}$$

Boundary condition (7) in the problem, Eqs. (9)–(11), is absent, since it does not affect the process of heat transfer in its first stage.

In view of the fact that the quantity $q_1(x)$ has been introduced into consideration, we must add conditions fulfilled at the TPF. In the given case these are the conditions of thermal insulation of the moving interface:

$$\Theta (q_1, x) = 0 , \quad \left. \frac{\partial \Theta (z, x)}{\partial z} \right|_{z=q_1} = 0 . \tag{12}$$

The first relation in (12) indicates that the temperature of the liquid on the moving interface is equal to its temperature at the channel inlet, and the second relation implies the absence of a heat flux beyond the TPF. A mathematical proof of the conditions of thermal insulation of the moving interface is given in [10] (Eq. (2.76)).

Now we impose the requirement that the sought solution of the problem, Eqs. (9)–(12), could satisfy the thickness-averaged thermal layer $0 \le z \le q_1(x)$ rather than the initial equation (9). Determining the integrals of the right- and left-hand sides of Eq. (9) over z within the limits from zero to $q_1(x)$, we find the following integral condition (the integral of heat balance):

$$\int\limits_0^{q_1(x)} (2z - z^2) \frac{\partial \Theta (z, x)}{\partial x} dz = \int\limits_0^{q_1(x)} \frac{\partial}{\partial z} \left(\frac{\partial \Theta (z, x)}{\partial z} \right) dz.$$

Having determined the integral on the right-hand side, we obtain

$$\int\limits_0^{q_1(x)} (2z - z^2) \frac{\partial \Theta (z, x)}{\partial x} dz = \left. \frac{\partial \Theta (z, x)}{\partial z} \right|_0^{q_1(x)} = \left. \frac{\partial \Theta (z, x)}{\partial z} \right|_{z=q_1} - \left. \frac{\partial \Theta (z, x)}{\partial z} \right|_{z=0} .$$

Subject to the second relation from (12) the heat balance integral takes the form

$$\int\limits_0^{q_1(x)} (2z - z^2) \frac{\partial \Theta (z, x)}{\partial x} dz = - \left. \frac{\partial \Theta (z, x)}{\partial z} \right|_{z=0} . \tag{13}$$

Let us represent the unknown temperature profile in the form of a polynomial:

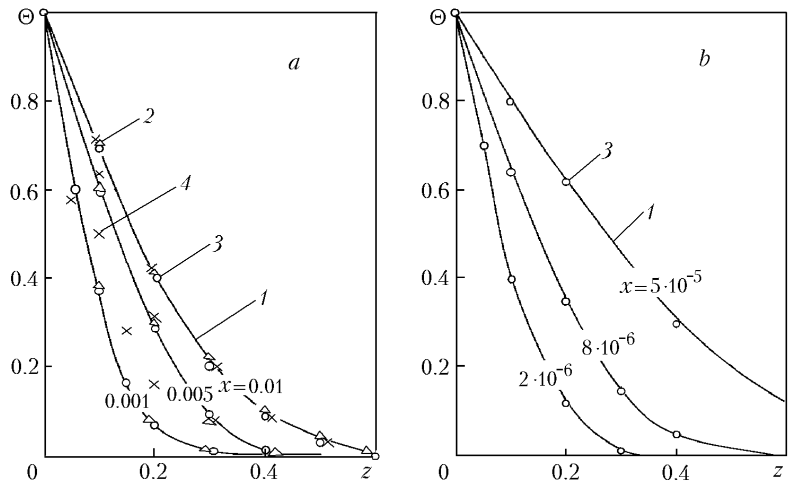

Fig. 2. Change in the dimensionless temperature of the liquid in a plane-parallel channel: 1) calculation by Eq. (26) (second approximation); 2) calculation by Eq. (15) (first approximation); 3) pivot method; 4) exact solution [8].

$$\Theta(z, x) = \sum_{k=0}^{n} a_k(q_1)\, z^k . \tag{14}$$

Substituting (14), limiting ourselves to three terms of the series, into boundary conditions (11) and (12), we obtain a system of three algebraic linear equations to determine the coefficients a_k (k = 0, 1, 2). Its solution is

$$a_0 = 1 , \quad a_1(q_1) = -2/q_1 , \quad a_2(q_1) = 1/q^2 .$$

The use of the found values of the coefficients a_k (k = 0, 1, 2) in (14) yields

$$\Theta(z, x) = \left(1 - \frac{z}{q_1}\right)^2 . \tag{15}$$

Substituting (15) into the heat balance integral (13), we have

$$\frac{\partial}{\partial x} \int_0^{q_1(x)} (2z - z^2) \left(1 - \frac{z}{q_1}\right)^2 dz = \frac{2}{q_1} .$$

Determining the integral on the left-hand side of the last relation for the unknown function $q_1(x)$, we arrive at the following ordinary differential equation:

$$\frac{\partial}{\partial x}\left(\frac{1}{6} q_1^2 - \frac{1}{30} q_1^3\right) = \frac{2}{q_1} .$$

Separating the variables and integrating, under the initial condition $q_1(0)$ = 0, we obtain

$$x = \frac{q_1^3}{2}\left(\frac{1}{9} - \frac{q_1}{40}\right). \tag{16}$$

Relations (15) and (16) determine the solution of the problem, Eqs. (9)–(12), in the first approximation. Assuming q_1 = 1, we find the distance traversed by the liquid along the axis x when the temperature perturbation front researches the center of the plane-parallel channel $x = x_1 = 0.038888$. The values of $q_1(x)$ obtained from Eq. (16) for different x are

x	0.001	0.0025	0.005	0.0075	0.01
$q_1(x)$	0.267555	0.366027	0.464955	0.5354196	0.592186

The results of calculations of the dimensionless temperature from Eq. (15) with allowance for the given data in comparison with the exact solution (Eq. (6–37) from [8]) as well as with the calculation by the pivot method are presented in Fig. 2. In the exact solution eight terms of the series were taken. The use of such a small number of the series terms of the exact solution is due to the complexity of the solution, obtained in [8] for the algebraic polynomials to determine eigennumbers, with the polynomial degree corresponding to the number of approximations. An analysis of the results allows the conclusion that the values of the temperatures calculated by Eq. (15) in the range of the dimensionless coordinate $0.01 \le x \le 0.001$ differ from the temperatures found with the aid of the pivot method by no more than 2%. At smaller values of the coordinate x the discrepancy increases. We note that the eight terms of the series of the exact solution from [8] prevent attainment of a high accuracy in determining temperatures at small values of x. In particular, at $x = 0.001$ (Fig. 2a) the results of the exact solution differ by 10–12% from the results of calculation by the pivot method. At the same time, the latter results for the coordinate in the range $0.01 \le x \le 2 \cdot 10^{-6}$ differ from those obtained by Eq. (26) in the second approximation by no more than 1%.

To obtain the solution of the problem, Eqs. (9)–(12), in the second approximation, we introduce supplementary boundary conditions. Using Eq. (9) for the point $z = 0$, we find the first supplementary boundary condition:

$$\left.\frac{\partial^2 \Theta(z, x)}{\partial z^2}\right|_{z=0} = 0 . \tag{17}$$

To obtain the second supplementary boundary condition, we differentiate the first relation in (12) with respect to x:

$$\frac{\partial \Theta(q_1, x)}{\partial x} + \frac{\partial \Theta(q_1, x)}{\partial z}\frac{dq_1}{dx} = 0 . \tag{18}$$

Taking into account the second condition from (12), we bring relation (18) to the form

$$\frac{\partial \Theta(q_1, x)}{\partial x} = 0 . \tag{19}$$

Applying Eq. (9) to $z = q_1(x)$ yields

$$\left.\frac{\partial \Theta(z, x)}{\partial x}\right|_{z=q_1} = \frac{1}{2q_1 - q_1^2}\left.\frac{\partial^2 \Theta(z, x)}{\partial z^2}\right|_{z=q_1} . \tag{20}$$

Comparing Eqs. (19) and (20), we find the second supplementary boundary condition:

$$\left.\frac{\partial^2 \Theta(z, x)}{\partial z^2}\right|_{z=q_1} = 0 . \tag{21}$$

To determine the third supplementary boundary condition, we differentiate the second relation from (12) with respect to x:

$$\left.\frac{\partial^2\Theta(z,x)}{\partial x\partial z}\right|_{z=q_1}=0\,. \tag{22}$$

We will differentiate Eq. (9) with respect to z and write it for $z = q_1(x)$:

$$\left[(2-2z)\frac{\partial\Theta(z,x)}{\partial x}\right]_{z=q_1}+\left[(2z-z^2)\frac{\partial^2\Theta(z,x)}{\partial x\partial z}\right]_{z=q_1}=\left.\frac{\partial^3\Theta(z,x)}{\partial z^3}\right|_{z=q_1}. \tag{23}$$

Relation (23) subject to Eq. (19) takes the form

$$\left.\frac{\partial^2\Theta(z,x)}{\partial x\partial z}\right|_{z=q_1}=\frac{1}{2q_1-q_1^2}\left.\frac{\partial^3\Theta(z,x)}{\partial z^3}\right|_{z=q_1}. \tag{24}$$

Comparing Eqs. (22) and (24), we find the third supplementary boundary condition:

$$\left.\frac{\partial^3\Theta(z,x)}{\partial z^3}\right|_{z=q_1}=0\,. \tag{25}$$

In the second approximation the assigned boundary conditions (11) and (12) and supplementary conditions (17), (21), and (25) allow us to determine already six coefficients of the temperature profile, Eq. (14). Substituting Eq. (14) into all basic and supplementary boundary conditions, we obtain a system of six algebraic linear equations to determine the coefficients a_k ($k = \overline{0,5}$). Its solution is

$$a_0=1\,,\quad a_1=-\frac{5}{2q_1}\,,\quad a_2=0\,,\quad a_3=\frac{5}{2q_1^3}\,,\quad a_4=-\frac{5}{2q_1^4}\,,\quad a_5=\frac{3}{2q_1^5}\,.$$

Substituting the found values of the coefficients a_k ($k = \overline{0,5}$) into (14), we find

$$\Theta(z,x)=1-\frac{5}{2}\frac{z}{q_1}+5\frac{z^3}{q_1^3}-5\frac{z^4}{q_1^4}+\frac{3}{2}\frac{z^5}{q_1^5}\,. \tag{26}$$

Substituting (26) into the heat balance integral (13), for the function $q_1(x)$ we arrive at the following ordinary differential equation:

$$\frac{2}{5}q_1\left(\frac{4}{21}q_1-\frac{5}{112}q_1^2\right)dq_1=dx\,. \tag{27}$$

Separating the variables and integrating, under the initial condition $q_1(0) = 0$, we obtain

$$\frac{15}{4}q_1^4-\frac{64}{3}q_1^3=-840x\,. \tag{28}$$

The values of $q_1(x)$, found by Eq. (28), for different x are

x	0.02093	0.01	0.005	0.001	0.0005	0.0001
$q_1(x)$	0.999956	0.769320	0.603916	0.347476	0.274509	0.159412

Assuming in Eq. (28) that $q_1 = 1$, we determine the time of the end of the first stage of the process in the second approximation $x = x_1 = 0.02093254$.

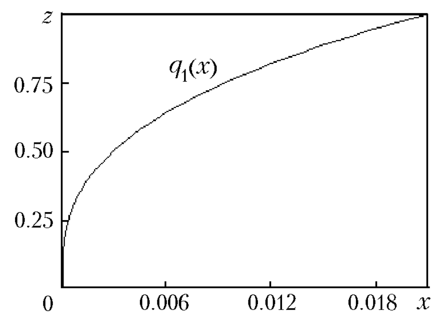

Fig. 3. Displacement of the temperature perturbation front over the coordinate z depending on the path traversed by the liquid over the coordinate x.

The results of calculations by Eq. (26) with allowance for the above-given data and in comparison with the exact solution from [8] as well as with the calculation by the pivot method are presented in Fig. 2. An analysis of these results allows the conclusion that within the range $0.01 \le x \le 2 \cdot 10^{-6}$ the difference between the temperature data obtained in the present work and the results of the pivot method does not exceed 1%.

Figure 3 presents the graph of the dependence of the temperature perturbation front on the dimensionless coordinate x. Analyzing this graph, we may see that the displacement of the temperature perturbation front along the coordinate z is most intense over a very small initial stretch of the coordinate x. For example, when the temperature perturbation front attains the distance $\Delta z = \Delta y = 0.15$, the value of x is $8.345 \cdot 10^{-5}$. Let us find an average velocity of the displacement of the temperature perturbation front over the stretch of the coordinate $0 \le z \le 0.15$ at the following initial data: $h = 0.01$ m, $\omega_{av} = 0.1$ m /sec, and $a = 14.3 \cdot 10^{-8}$ m^2/sec. The dimensionless distance $x = 8.345 \cdot 10^{-5}$ in a dimensional form is determined as $\eta = 3 \text{ Pe } hx/8 = 1.094187 \cdot 10^{-3}$ m. The liquid covers this distance in the time equal to $\tau = 0.010941$ sec. The displacement of the temperature perturbation front over the coordinate z in a dimensional form is $\Delta\xi = \Delta yr = 0.00425$ m. From this the average velocity of the displacement of the temperature perturbation front over the coordinate stretch $0 \le \xi \le 0.00425$ m in time $\tau = 0.010941$ sec is $v_{av} = \Delta\xi/\tau = 0.3885$ m/sec. At $z = 0.05$ ($\xi = 0.00475$ m) $x = 3.1467 \cdot 10^{-6}$ ($\eta = 4.1259 \cdot 10^{-5}$ m) and $\tau = 4.1259 \cdot 10^{-4}$ sec. The average velocity v_{av} over the stretch $0 \le z \le 0.05$ is equal to 11.51 m/sec. From this it follows that with a decrease in the considered range over the coordinate z the average velocity of the temperature perturbation front increases substantially.

Supplementary boundary conditions for the third approximation are obtained similarly. For this purpose, we differentiate boundary conditions (11), (21), and (25) with respect to the variable x:

$$\frac{\partial \Theta (0, x)}{\partial x} = 0 , \tag{29}$$

$$\left. \frac{\partial^3 \Theta (z, x)}{\partial z^2 \partial x} \right|_{z=q_1} = 0 , \tag{30}$$

$$\left. \frac{\partial^4 \Theta (z, x)}{\partial z^3 \partial x} \right|_{z=q_1} = 0 . \tag{31}$$

Differentiating Eq. (9) with respect to z and applying it to the point $z = 0$, subject to (29), we obtain the following supplementary boundary condition:

$$\left. \frac{\partial^3 \Theta (z, x)}{\partial z^3} \right|_{z=0} = 0 . \tag{32}$$

Having differentiated Eq. (9) twice with respect to z and applied it to the point $z = q_1$, subject to (19) and (22), we have

$$\left.\frac{\partial^3\Theta(z,x)}{\partial z^2\partial x}\right|_{z=q_1} = \frac{1}{2q_1 - q_1^2}\left.\frac{\partial^4\Theta(z,x)}{\partial z^4}\right|_{z=q_1} . \tag{33}$$

Comparing Eqs. (30) and (33), we find the following supplementary boundary condition:

$$\left.\frac{\partial^4\Theta(z,x)}{\partial z^4}\right|_{z=q_1} = 0 . \tag{34}$$

Having differentiated Eq. (9) thrice with respect to z and applied it to the point $z = q_1$, subject to (22) and (30), we obtain

$$\left.\frac{\partial^4\Theta(z,x)}{\partial z^3\partial x}\right|_{z=q_1} = \frac{1}{2q_1 - q_1^2}\left.\frac{\partial^5\Theta(z,x)}{\partial x^5}\right|_{z=q_1} . \tag{35}$$

Comparing (31) and (35), we write one other supplementary boundary condition

$$\left.\frac{\partial^5\Theta(z^5,x)}{\partial z^5}\right|_{z=q_1} = 0 . \tag{36}$$

The basic boundary conditions (11) and (12) and supplementary conditions (17), (22), (25), (32), (34), and (36) allow one to determine already nine coefficients of the temperature profile, Eq. (14), and to obtain the solution of the problem (Eqs. (9)–(12)) in the third approximation.

The method of supplementary boundary conditions based on the heat balance integral can be applied also to the second stage of the heat transfer process. This stage, which corresponds to the longitudinal coordinate $x \ge x_1$ is characterized by a change in the temperature already over the entire section of the channel up to the state in which the whole liquid is heated up to the wall temperature. For this stage the notion of a thermal layer loses its meaning, and as a generalized coordinate one selects a temperature at the channel center $q_1 = q_2(x)$ (see Fig. 1b).

The mathematical statement of the problem for the second stage of the process has the form

$$(2z - z^2)\frac{\partial\Theta(z,x)}{\partial x} = \frac{\partial^2\Theta(z,x)}{\partial z^2} , \quad x \ge x_1 , \quad 0 \le z < 1 ; \tag{37}$$

$$\Theta(z, x_1) = f(z) ; \tag{38}$$

$$\Theta(0,x) = 1 , \quad \Theta(1,x) = q_2(x) , \quad \left.\frac{\partial\Theta(z,x)}{\partial z}\right|_{z=1} = 0 , \tag{39}$$

where $f(z)$ is the solution for the first stage of the process at $x = x_1$. Averaging differential equation (37) over the entire volume of the medium $(0 \le z \le 1)$, we obtain a heat balance integral of the form

$$\int_0^1 (2z - z^2)\frac{\partial\Theta(z,x)}{\partial x}dz = -\frac{\partial\Theta(0,x)}{\partial z} . \tag{40}$$

The solution which satisfies integral condition (40) is sought in the form of a polynomial:

$$\Theta(z, x) = \sum_{k=0}^{n} b_k(q_2) z^k . \tag{41}$$

Let us find the solution of the problem, Eqs (37)–(39), in the first approximation. As the initial condition we use relation (15) at $q_1(x_1) = 1$ which is the solution of problem (9)–(12) in the first approximation of the first stage of the process. Thus, initial condition (38) in the given case takes the form

$$\Theta(z, x_1) = (1 - z)^2 . \tag{42}$$

Substituting (41), limiting ourselves to the three terms of the series, into boundary conditions (39), we obtain a system of three algebraic equations to determine the coefficients b_k ($k = 0, 1, 2$). Having found b_k, we substitute them into (41) and obtain

$$\Theta(z, x) = 1 - z(1 - q_2)(2 - z) . \tag{43}$$

Substituting (43) into the heat balance integral (40), for the unknown function $q_2(x)$ we obtain the following ordinary differential equation:

$$\frac{8}{15}\frac{d}{dx}(1 - q_2) = -2(1 - q_2) . \tag{44}$$

Separating the variables and integrating, we find

$$\ln|1 - q_2| = -\frac{15}{4}x + C , \tag{45}$$

where $C = 15x_1/4$ is the integration constant determined from the initial condition $q_2(x_1) = 0$. Relation (45) with allowance for the value found for the integration constant has the form

$$q_2 = 1 - \exp\left[-\frac{15}{4}(x - x_1)\right]. \tag{46}$$

Substituting (46) into (43), we obtain the solution of problem (37)–(39) in the first approximation:

$$\Theta(z, x) = 1 - z(2 - z)\exp\left[-\frac{15}{4}(x - x_1)\right], \tag{47}$$

where for x_1 we take the value 0.038888 obtained from Eq. (16) in the first approximation of the first stage of the process at $q_1(x_1) = 1$. Using direct substitution, we can see that relation (47) rather accurately satisfies the heat balance integral (40), initial condition (42), and boundary conditions (39).

The results of calculations by Eq. (47) as compared to the exact solution of [8] as well as to the calculation by the pivot method are presented in Fig. 4. Analyzing them, we can conclude that at values of the variable x slightly differing from x_1 ($0.08 \le x \le 0.15$), the discrepancy between the values obtained in the present work and the exact solution lies within 6–8%. As the coordinate x increases, the discrepancy between the temperatures increases up to 12%.

To raise the accuracy, we will find the solution of the problem, Eqs. (37)–(39), in the second approximation involving supplementary boundary conditions. To find the latter, we will differentiate the second and third relations from (39) with respect to the variable x:

$$\left.\frac{\partial\Theta(z, x)}{\partial x}\right|_{z=1} = \frac{\partial q_2(x)}{\partial x} , \quad \left.\frac{\partial^2\Theta(z, x)}{\partial z\partial x}\right|_{z=1} = 0 . \tag{48}$$

Using Eq. (37) for the point $z = 0$, we obtain the first supplementary boundary condition:

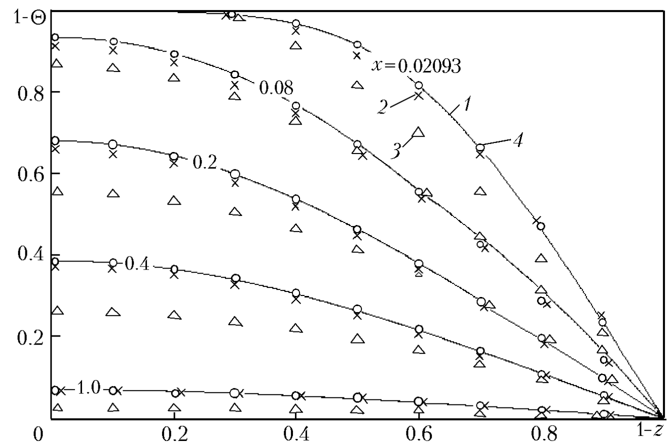

Fig. 4. Change in the dimensionless temperature of the liquid in a plane-parallel channel: 1) exact solution [8]; 2) calculation by Eq. (59) (second approximation); 3) calculation by Eq. (47) (first approximation); 4) pivot method.

$$\left.\frac{\partial^2\Theta(z,x)}{\partial z^2}\right|_{z=0} = 0 . \tag{49}$$

Equation (37) for the point $z = 1$ is

$$\left.\frac{\partial\Theta(z,x)}{\partial x}\right|_{z=1} = \left.\frac{\partial^2\Theta(z,x)}{\partial z^2}\right|_{z=1} . \tag{50}$$

Comparing the first relation from (48) with relation (50), we find the second supplementary boundary condition:

$$\left.\frac{\partial^2\Theta(z,x)}{\partial z^2}\right|_{z=1} = \frac{\partial q_2(x)}{\partial x} . \tag{51}$$

To obtain the third supplementary boundary condition we differentiate Eq. (37) with respect to the variable z and write it for the point $z = 1$:

$$\left.\frac{\partial^2\Theta(z,x)}{\partial x\partial z}\right|_{z=1} = \left.\frac{\partial^3\Theta(z,x)}{\partial z^3}\right|_{z=1} . \tag{52}$$

Comparing the second relation from (48) with relation (52), we obtain the third supplementary boundary condition:

$$\left.\frac{\partial^3\Theta(z,x)}{\partial z^3}\right|_{z=1} = 0 . \tag{53}$$

The basic boundary conditions (39) and supplementary conditions (49), (51), and (53) allow one to determine as many as six coefficients of the temperature profile (41). Substituting (41) into the listed boundary conditions for the unknown quantities $b_k(q_2)$ ($k = \overline{0, 5}$), we obtain a system of six algebraic linear equations:

$$\left(b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + b_5 z^5\right)_{z=0} = 1 , \quad b_0 + b_1 + b_2 + b_3 + b_4 + b_5 = q_2 ,$$

$$b_1 + 2b_2 + 3b_3 + 4b_4 + 5b_5 = 0 , \quad \left(2b_2 + 6b_3 z + 12b_4 z^2 + 20b_5 z^3\right)_{z=0} = 0 , \tag{54}$$

$$2b_2 + 6b_3 + 12b_4 + 20b_5 = \frac{dq_2}{dx}\,, \quad 6b_3 + 24b_4 + 60b_5 = 0\,.$$

From the first and fourth equations of system (54) it follows that $b_0 = 1$ and $b_2 = 0$. The solution of the system of equations (54) for the remaining coefficients has the form

$$b_1 = \frac{5}{2}(q_2 - 1) + \frac{3}{8}\frac{\partial q_2}{\partial x}\,, \; b_3 = 5(1 - q_2) - \frac{7}{4}\frac{\partial q_2}{\partial x}\,, \; b_4 = 5(q_2 - 1) + 2\frac{\partial q_2}{\partial x}\,, \; b_5 = \frac{3}{8}(1 - q_2) - \frac{5}{8}\frac{\partial q_2}{\partial x}\,. \tag{55}$$

Substituting (41), subject to the values obtained for the coefficients $b_k(q_2)$ ($k = \overline{0, 5}$), into the heat balance integral (40), for the unknown function $q_2(x)$ we obtain the following second-order ordinary differential equation:

$$\frac{\partial^2 q_2}{\partial x^2} + \frac{6460}{191}\frac{\partial q_2}{\partial x} + \frac{16{,}800}{191} q_2 - \frac{16{,}800}{191} = 0\,. \tag{56}$$

The general integral of Eq. (56) has the form

$$q_2(x) = 1 + C_1 \exp(2.8389x) + C_2 \exp(30.9831x)\,, \tag{57}$$

where C_1 and C_2 are the integration constants determined from the boundary conditions

$$q_2(x_1) = 0\,, \quad \left.\frac{\partial q_2(x)}{\partial x}\right|_{x=x_1} = 0\,. \tag{58}$$

Substitution of (57) into (58) yields

$$C_1 = -1.100870 \exp(2.8389x_1)\,, \quad C_2 = 0.100870 \exp(30.9831x_1)\,.$$

Equation (41), subject to Eq. (57) and the values found for the coefficients C_1 and C_2, takes the form

$$\Theta(x, z) = 1 - \left(A_1 z - A_3 z^3 - A_5 z^4 + A_7 z^5\right)\exp\left[-\nu_1 (x - x_1)\right] -$$
$$- \left(A_2 z - A_4 z^3 + A_6 z^4 - A_8 z^5\right)\exp\left[-\nu_2 (x - x_1)\right], \tag{59}$$

where $\nu_1 = 2.8389$; $\nu_2 = 30.9831$; $A_1 = 1.5802$; $A_2 = 0.9198$; $A_3 = 0.0351$; $A_4 = 4.9648$; $A_5 = 0.7462$; $A_6 = 5.7462$; $A_7 = 0.3020$; $A_8 = 1.8020$, and $x_1 = 0.02093254$. The value of x_1 was obtained from Eq. (28) in the second approximation of the first stage of the process at $q_1 = 1$.

We note that the coefficients ν_1 and ν_2 under the exponential sign differ slightly from the first two eigennumbers of the boundary-value problem, Eqs. (5)–(8), in the case where the classical analytical method of the separation of variables is used to solve it. Their exact values [8] are: $\nu_1 = 2.8278$ and $\nu_2 = 32.1473$.

The results of calculations by Eq. (41) in comparison with the exact solution from [8] as well as with calculation by the finite-difference pivot method are presented in Fig. 4. Analyzing them, we may conclude that the values of temperature obtained by Eq. (59) within the range $x_1 \le x \le \infty$ differ from their precise values by no more than 2%. We note that the temperatures obtained by the pivot method in this very range of the variable x practically coincide with their precise values. As indicated above, eight terms of the series were used in the exact solution.

CONCLUSIONS

1. Based on the integral heat balance method, using the notion of the temperature perturbation front and supplementary boundary conditions, analytical solutions of the problems of heat transfer in a stabilized liquid flow in a tube have been obtained. The process of heat transfer is divided into two stages. The first includes the range over the

coordinate $0 \le x \le x_1$ within which the temperature perturbation front, while moving from the surface, reaches the center of the channel. In the second stage the process of heat transfer occupies the entire width of the channel $0 \le z \le 1$ within the range $x_1 \le x < \infty$.

2. A technique to obtain supplementary boundary conditions is suggested which is based on using the initial differential equation and given boundary conditions, including the conditions at the temperature perturbation front. Using them, it is possible, with a minimum number of approximations, to considerably improve the fulfillment of the basic differential equation in the entire range of variation of the variable z $(0 \le z \le 1)$, since due to the satisfaction of the supplementary boundary conditions this equation is satisfied exactly at all the points where the temperature perturbation front is present at the corresponding values of the variable x.

NOTATION

a, thermal diffusivity coefficient, m^2/sec; $a_k(q_1)$ and $b_k(q_2)$, unknown coefficients; $h = 2r$, channel width, m; Pe, Peclet number; r, half width of the plane channel, m; t, temperature, ^oC; t_0, temperature at the inlet to the channel, ^oC; t_w, wall temperature, ^oC; x, dimensionless longitudinal coordinate; y, dimensionless transverse coordinate; η, longitudinal coordinate, m; Θ, relative excess temperature; ξ, transverse coordinate, m; τ, time, sec; $\omega = \frac{3}{2}\omega_{av}\left(1 - \frac{\xi^2}{r^2}\right)$, distribution of the liquid velocity over the coordinate ξ $(0 \le \xi \le r)$, m/sec; ω_{av}, average velocity, m/sec. Subscripts: w, parameters on the wall; 0, parameters at the inlet to the channel; av, average values.

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